A Note on the Equilibrium of the OTC Game

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As it was noticed by Yilin Wang, in our paper Babus and Kondor (2018) the first-order condition (10) is not fully consistent with our description of the OTC game. In particular, we derive the first order condition for dealer i from problem

$$\max_{(q^i_{ij})_{j\in g^i}} \sum_{j\in g^i} q^i_{ij}\left(E\left(\theta^i|s^i,\mathbf{p}_{g^i}\right)+\frac{1}{\beta_{ij}+t^j_{ij}\left(z^j_{ij,ij}-1\right)}q^i_{ij}-I^j_{ij}\right)$$

where we take the intercept I^j_{ij} defined by (8) as insensitive to the quantity q^i_{ij}.

In our derivations, we implicitly assume that each dealer i chooses his demand function when trading with a neighbor $j \in g^i$ to maximize her objective function (2) understanding that his residual demand on link ij is determined by market clearing conditions

$$Q^i_{ij}\left(s^i,p_{g^i}\right)+Q^j_{ij}\left(s^j,p_{g^j}\right)+\beta_{ij}p_{ij}=0,$$

but taking all other prices as given.

However, while dealer i does not observe prices $p_{jk,k\neq i}$, at which she does not trade, in a Bayesian Nash equilibrium dealer i should still consider the indirect effect of her quantity q^i_{ij}on p_{ij} through p_{jk}.

In this corrigendum, we are proceeding as follows. First, we introduce a modification of the OTC game which restores consistency and leaves all our results and proofs unchanged. Second, returning to the original specification, we discuss a family of networks where the indirect price effect leaves our qualitative results virtually unchanged. We also state open questions for future research.

1 The modified OTC game: Dealerships instead of dealers

Instead of a game with n dealers, consider n groups of dealers organized into the trading network, g. Each group i contains a mass of $m^i \equiv \left|g^i\right|$ dealers who observe the same private signal s^i, have the same valuation for the asset, θ^i, and share all the relevant information on their transaction prices within the group as we specify below. Each group i is divided into m^i unit-mass subgroups and each subgroup is allocated to trade at a single link only. The joint distribution of all random variables are as in Babus and Kondor (2018). We show that, apart from more cumbersome notation, this specification is very similar to Babus and Kondor (2018). With this modification all the results and proofs go through without any change.

Within each subgroup, we index dealers by $\tau \in [0,1]$. Therefore, a particular dealer is identified by its group, the link at which she trades at, and her index in the subgroup, (i, ij, τ), where $j \in g^i$. Without loss of generality, we assume that the counterparty of dealer (i, ij, τ) is dealer (j, ij, τ). These two dealers trade at price $\hat{p}_{ij}(\tau)$. Let the trading strategy of dealer (i, ij, τ) be a demand function

$$Q^i_{ij,\tau}\left(s^i, \mathbf{p}_{g^i}, \hat{p}_{ij}(\tau)\right) \tag{C1}$$

which maps the signal of the group, s^i, the vector of average prices, $\mathbf{p}_{g^i}$, that prevail in the group's transactions and the transaction price $\hat{p}_{ij}(\tau)$ into a traded quantity. Let us denote this quantity by $\hat{q}^i_{ij}(\tau)$. In particular, elements of $\mathbf{p}_{g^i}$ are defined as

$$p_{ik} \equiv \int_0^1 \hat{p}_{ik}(\tau)\, d\tau$$

for all $k \in g^i$. Dealer (j, ij, τ) chooses a strategy to maximize her expected profit

$$E\left(Q^i_{ij,\tau}\left(s^i, \mathbf{p}_{g^i}, \hat{p}_{ij}(\tau)\right)\left(\theta^i - \hat{p}_{ij}(\tau)\right)|s^i, \mathbf{p}_{g^i,k\neq j}, \hat{p}_{ij}(\tau)\right)$$

where $\hat{p}_{ij}(\tau)$ is a function which is continuous almost everywhere, and defined as the solution of

$$Q^i_{ij,\tau}\left(s^i, \mathbf{p}_{g^i}, \hat{p}_{ij}(\tau)\right) + Q^j_{ij,\tau}\left(s^j, \mathbf{p}_{g^j}, \hat{p}_{ij}(\tau)\right) + \beta_{ij}\hat{p}_{ij}(\tau) = 0$$

for every link ij and index τ.[1] Just as in Babus and Kondor (2018), β_{ij} corresponds to the representative share of customers at the given link.

Then, Babus and Kondor (2018) finds a symmetric Linear Bayesian Nash equilibrium of this game where (1) $\hat{q}^i_{ij}(\tau) \equiv q^i_{ij}, \hat{p}_{ij}(\tau) \equiv p_{ij}$ are invariant in τ, (2) $Q^i_{ij,\tau}\left(s^i, \mathbf{p}_{g^i}, \hat{p}_{ij}(\tau)\right)$ are invariant in τ,hence have the form of $Q^i_{ij}\left(s^i, \mathbf{p}_{g^i}\right)$,

[1] If such $\hat{p}_{ij}(\tau)$ does not exist for all links and index, just as in the paper, we consider that markets break down and assign zero utility to all players. If there is more than one such group of functions, we choose by an arbitrary selection mechanism.

and (3) $\mathbf{Q}^i(s^i;\mathbf{p}_{g^i})$ is the collection of functions $Q^i_{ij}\left(s^i,\mathbf{p}_{g^i}\right)$.[2]

To see that with this modification our analysis remains intact and the indirect price effect disappears, note that the impact of any dealer (i,ij,τ) on the average price p_{ij} is infinitesimal and only that average price affects other dealers' beliefs.

2 Would the indirect price effect change the qualitative results?

In this part, we return to the Linear Bayesian Nash equilibrium of the unmodified game in Babus and Kondor (2018). We analyze the consequences of the indirect price effect in our main example, the $n-$star network. For this network, we can still derive the equilibrium in simple closed form expressions. We use the same notation as in Appendix B of Babus and Kondor (2018), and consider that dealer 1 is the centre, and dealers $2,...,n$ are the periphery. All our expressions in Appendix B remain intact, except the last three which modify to

$$z_C = \frac{\bar{z}_C 2}{1+(n-2)\,\bar{z}_C\,(1-\bar{z}_P)} \tag{C2}$$

$$y_C = \bar{y}_C\left(1-\frac{1}{2}\frac{z_C\,(n-1)\,(1-\bar{z}_P)\,(2-(n-2)\,z_C)}{2-(n-2)\,z_C-\bar{z}_P z_C}\right) \tag{C3}$$

$$y_P = \bar{y}_P\left(1-\bar{z}_P\frac{2-(n-1)\,z_C}{2-(n-2)\,z_C-\bar{z}_P z_C}\right). \tag{C4}$$

To see this, without loss of generality, we first derive the total price effect of periphery dealer n by solving the equation system

$$q^n_{n1}+t_C\left(y_C s^1+\Sigma^n_{k=2} z_C p_{1k}-p_{1n}\right)+\beta p_{1n}=0$$
$$t_P\left(y_P s^j+z_P p_{1j}-p_{1j}\right)+t_C\left(y_C s^1+\Sigma^n_{k=2} z_C p_{1k}-p_{1j}\right)+\beta p_{1j}=0.$$

for each $j=2..n-1$. The first equation gives

$$\frac{q^n_{n1}+t_C\left(y_C s^1+\Sigma^{n-1}_{j=2} z_C p_{1j}\right)}{t_C\,(1-z_C)-\beta}=p_{1n} \tag{C5}$$

while summing up the rest of the equations gives

$$\frac{t_P y_P \Sigma^{n-1}_{j=2} s^j+t_C\left((n-2)\,y_C s^1+(n-2)\,z_C p_{1n}\right)}{-\beta+t_C\,(1-(n-2)\,z_C)+t_P\,(1-z_P)}=\Sigma^{n-1}_{j=2} p_{1j}. \tag{C6}$$

[2] Note that under this specification each dealer (i,ij,τ) might observe two informationally equivalent signals: the transaction price $\hat{p}_{ij}(\tau)$, and the average transaction price p_{ij}. The equilibrium in Babus and Kondor (2018) is obtained by conjecturing that each dealer uses only $\hat{p}_{ij}(\tau)$ for inference and verifying that there is no profitable deviation. An alternative is to directly restrict the strategy space to demand curves of the form

$$Q^i_{ij,\tau}\left(s^i,\mathbf{p}_{g^i,k\neq j},\hat{p}_{ij}(\tau)\right)$$

instead of (C1).

Combining (C5)-(C6), we obtain that the inverse residual demand curve is given by

$$p_{1n} = I^1_{n1} + \lambda^1_{n1} q^n_{n1}$$

with

$$\lambda^1_{n1} = \frac{1}{t_C\left(1-z_C\right)-\beta} \frac{1}{\left(1-\frac{(z_C)^2(t_C)^2(n-2)}{(-\beta+t_C(1-(n-2)z_C)+t_P(1-z_P))(t_C(1-z_C)-\beta)}\right)}$$

Then, the first order condition for periphery dealer n modifies to

$$q^n_{n1} =$$

$$= \left(\left(t_C\left(1-z_C\right)-\beta\right) - \frac{\left(t_C\left(1-z_C\right)-\beta\right)\left(z_C\right)^2\left(t_C\right)^2\left(n-2\right)}{\left(-\beta+t_C\left(1-\left(n-2\right)z_C\right)+t_P\left(1-z_P\right)\right)\left(t_C\left(1-z_C\right)-\beta\right)}\right)\left(E\left(\theta^n|s^n,p_{1n}\right)-p_{1n}\right). \tag{C7}$$

The form of the first order condition of the central agent still implies (12). Then, solving for t_C and t_P, we get

$$t_P = -2\beta\frac{\left(n-1\right)z_C-2}{nz_Pz_C-2z_P-2z_C} \tag{C8}$$

$$t_C = -\beta\left(2-z_P\right)\frac{\left(n-2\right)z_C-2}{nz_Pz_C-2z_P-2z_C} \tag{C9}$$

which need to be compared to equation (22) in the paper.

Importantly, (C7) shows that, even when we account for the indirect price effect, the demand function of dealer j still has the form of $t_P\left(E\left(\theta|s^j,p_{1j}\right)-p_{1j}\right)$. This implies that the counterparty of agent j can learn the posterior of agent j from the market clearing price. Similarly, any periphery dealer j can learn the belief of the central dealer from the market clearing price. This critical property allows us to follow Proposition 2 and use the conditional guessing game to derive the equilibrium. In the particular case of the n-star, instead of system (21), we use $\bar{z}_P = \frac{1}{2}z_P$ from Appendix B and express $\bar{z}_C, \bar{y}_P, \bar{y}_C$ from (C2)-(C4). Given these equations, following the proof of Proposition 2 we show that choosing the prices and demand functions (23) and (24) in Babus and Kondor (2018) is an equilibrium of the OTC game in the n-star network, where the trading intensities are given by (C8)-(C9).

Note also, that our general observations in Section 5.1.1 relied only on this property and that (22) implies $\frac{\partial t^i_{ij}}{\partial z^j_{ij}} < 0$, which still holds under (C8)-(C9). We have also checked that the other statements concerning the n-star network, the second part of Proposition 3 and the first part of Proposition 9 still holds unchanged.[3]

[3] As for the second part of Proposition 9, while z_C, z_P, y_C, y_P converges to the same values in the limit $\rho \to 1$ as in Babus and Kondor (2018), because of the changing expressions of (C8)-(C9), the limits of t_C and t_P change to $t_C \to -\frac{\beta}{n}$ and $t_P \to -(n-1)\frac{\beta}{n}$. In this limit, each dealer's profit is bounded away from zero. Details are available on request.

The effect of the indirect price effect on the equilibrium analysis is negligible for the $n-$star network. However, the characterization of the Bayesian Nash equilibrium for general networks of our unmodified game remains an open problem.

References

Babus, A., and P. Kondor (2018): "Trading and Information Diffusion in Over-the-Counter Markets," *Econometrica*, 86(5), 1727–1769.